The Role of Symmetry in Modern Physics

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I. INTRODUCTION

Perhaps the most impressive intellectual achievement in the history of mankind, the Standard Model of particle physics is the triumph of modern physics. Indeed, it is able to explain almost all physical phenomena that we have ever encountered with astonishing accuracy. The Standard Model is technically classified as a quantum field theory (QFT) - a mathematical framework that quantizes the classical notion of a field while taking (special) relativistic corrections into account. It is composed of two subtheories, the electroweak theory (EWT) and quantum chromodynamics (QCD).

The electroweak theory is a unified theory that quantum mechanically describes both electrodynamics and the weak force. Electrodynamics describes the interactions between particles with electric charge while the theory of the weak force describes processes such as radioactive decay (interactions involving weak isospin). Although these two forces seem to be completely distinct, it has been shown that at high energies they combine into a single, more fundamental electroweak force. This remarkable discovery not only marked the birth of the Standard Model, but also supported the long-standing belief that there may exist a single theory that can explain everything in our universe. Quantum chromodynamics, on the other hand, is the theory of the strong interaction - it describes how quarks are held together in hadrons and how nucleons are held together in atomic nuclei.

Note, however, that the gravitational interaction does not seem to be in this picture; this is due to the fact that gravity is simply not included in the Standard Model. Indeed, it is the Standard Model’s greatest weakness - no one has found a consistent method of integrating general relativity into a quantum field theoretic framework. Many theorists believe that this flaw points towards physics beyond the Standard Model, which has recently been substantiated with discoveries such as neutrino oscillations [1].

II. SYMMETRY

All of theoretical particle physics is based soundly on complicated mathematical structures. Here, I will focus on one of the most important aspects of these structures - symmetry. The existence of symmetries in physical theories plays a crucial role in unifying all of particle physics. What exactly is symmetry, though? A definition can be formulated as follows:

**Definition 1.** A physical system $S$ is said to be symmetric (or invariant) under a transformation $T$ if

$$S' \equiv T(S) = S$$

In words, a system is symmetric under an operation if the system remains unchanged when operated upon.

Additionally, it can be easily shown using elementary group theory that the set of transformations $X$ leaving the system $S$ invariant forms a group called the symmetry group of $S$ [2]. The existence of such symmetries in physical laws was perhaps first appreciated by Lorentz [3]. Lorentz realized that Maxwell’s laws of classical electrodynamics were unchanged under the following set of transformations (now called Lorentz transformations):

$$t' = \gamma(t - vx/c^2)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

with $$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

This set of equations describes how to convert between the measurements of two observers in different reference frames (specifically, what is called the standard configuration). However, it was not until Einstein’s discovery of special relativity that physicists realized that Lorentz invariance is a fundamental part of spacetime that governs the form of Maxwell’s equations - not the other way around. It was this way of thinking that set the basis for the mathematical structure of future theories.
III. NOETHER’S THEOREM

One of the most important results related to symmetry in physics is called Noether’s theorem and is as follows:

**Theorem 1.** Any continuous symmetry of the action of a physical system has a corresponding conservation law.

Thus, given the fact that the laws of physics are the same, no matter where an experiment is conducted, this theorem states that there exists a corresponding conserved quantity - in this case, momentum [4, 5].

Noether’s theorem is important enough to warrant a derivation - let us examine how it arises in a classical system with transformations that affect coordinates. First, assume the system can described by a Lagrangian

\[ L[q_i(t), \dot{q}_i(t)] \]

From normal Lagrangian mechanics, we know that the system travels through its \( n \)-dimensional configuration space on a path that minimizes the action \( \int L dt \). Since we have assumed that the symmetry is continuous, we are justified in operating an infinitesimal transformation upon the system. This tiny transformation now changes the system’s path through configuration space by a tiny amount

\[ \delta A = \sum_{i=1}^{n} \left. \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right|_{t_1}^{t_2} + \sum_{i=1}^{n} \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt, \]

a result that we know from the application of the calculus of variations. Recall that we know the system is invariant under the transformation - otherwise there wouldn’t be a symmetry. Thus, the physical system itself must be unchanged and \( \delta A \) must be zero. Moreover, the second term of the above equation must vanish, as a result of the Euler-Lagrange equations. Thus

\[ \sum_{i=1}^{n} \left. \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right|_{t_1}^{t_2} = 0, \]

and we obtain the conservation law

\[ \sum_{i=1}^{n} \left. \frac{\partial L}{\partial q_i} \delta q_i \right|_{t_1}^{t_2} = 0. \]

Noether’s theorem is particularly important in that it shows how conservation laws do not arise as fundamental laws of nature, but instead from the symmetries inherent in the laws of physics. For example, the conservation of energy occurs as a result of the time-translational symmetry of physics - the laws of physics do not change over time - and the conservation of angular momentum occurs as a result of the rotational invariance of physics.

IV. GAUGE THEORIES

Let us now examine the symmetries present in Maxwell’s four equations of electrodynamics (under vacuum conditions):

\[ \nabla \cdot E = 0, \quad \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} \]

\[ \nabla \cdot B = 0, \quad \nabla \times B = \frac{1}{c} \frac{\partial E}{\partial t} \]

Just by a superficial examination of the four equations, it is evident that the behavior and interactions of the electric field are quite similar to those of the magnetic field [6]. However, there is a deeper level of symmetry to be found when one notices that the vector equations of magnetism suggest (by theorems of vector calculus) the existence of a vector potential \( A \):

\[ B = \nabla \times A \]

It is important to note that the vector field \( A \) has no direct physical significance - it is not an observable, unlike the magnetic field \( B \). Thus, we can vary its value at every point in space, and as long as \( B \) doesn’t change, Maxwell’s equations will remain invariant [7]. Specifically, the transformation of the vector potential

\[ A \mapsto A' = A + \nabla \psi \]

coupled with a similar transformation of the scalar potential

\[ \Phi \mapsto \Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \]

is called a gauge transformation (for arbitrary \( \psi \) and \( \Lambda \)) and does not change Maxwell’s equations. The symmetry that arises from this gauge invariance leads to conservation of charge and is thus crucial to the theory of classical electrodynamics. Indeed, this symmetry is so important that the theory is known as a gauge theory, a term that is used to classify physical theories that exhibit such invariance.

Another theory that incorporates gauge invariance is quantum electrodynamics (QED) - the quantum field theory of electromagnetism. The action of
the theory, the rather complicated expression,

$$A = \int \bar{\psi} (i \hbar c \gamma^\mu \partial_\mu - mc^2) \psi \ d^4x,$$

turns out to be invariant under the gauge transformation $\psi \mapsto e^{i\theta} \psi$. In words, the imaginary component (or phase) of the quantized electromagnetic fields (or electron field) can be arbitrarily changed without changing any physical, observable quantities. Noether’s theorem, when applied to this gauge symmetry, yields the law of conservation of electric charge.

The symmetry group for QED is the circle group U(1) as it represents all complex numbers with absolute value 1 [8]. Different theories have different symmetry groups to describe their interactions. For example, the electroweak theory has the symmetry group SU(2)×U(1) and quantum chromodynamics exhibits the symmetry group SU(3), where SU(n) is the special unitary group - the group of $n \times n$ unitary matrices with determinant 1.

It actually turns out that QED is symmetric under a local gauge transformation - a transformation that changes the phase of the electron field at every point in space by a possibly distinct amount. This invariance, quite remarkably, leads to the existence of the electromagnetic interaction itself. This however, is rooted in advanced mathematics and is far beyond the scope of the present discussion - suffice it to say that gauge symmetry has become an integral component of modern physics.

V. DISCRETE SYMMETRIES

In addition to the continuous symmetries mentioned above, the laws of physics exhibit many discrete near-symmetries. Discrete symmetries are symmetries that arise from non-continuous transformations on the system. There are 3 important discrete near-symmetries that are arise from the following 3 transformations: charge conjugation, parity inversion, and time reversal.

The first transformation, charge conjugation, swaps each particle with its antiparticle equivalent. It turns out that the laws of electromagnetism are invariant under this C-symmetry. In effect, this is equivalent to the statement that if we flipped what we called positive and negative, the laws of electromagnetism would remain the same - a statement that is intuitively true. C-symmetry, however, does not apply to all interactions, which is why it is known as a near-symmetry. It does not hold, for example, when applied to the weak force due to the subtleties of the interaction.

The next transformation, P-symmetry, is a flip in the sign of the spatial coordinates,

$$P: (x, y, z) \mapsto (-x, -y, -z).$$

This transformation essentially changes the laws of the physics to their mirror images. Again, although one intuitively expects that this symmetry would hold, it is violated by certain weak interactions. These issues are analogous to how a rotating wheel and its reflection have angular momentum vectors pointing in the same direction, instead of the expected opposite direction.

Finally, T-symmetry flips the arrow of time,

$$T: t \mapsto -t.$$ This discrete symmetry, unlike the previous two, does not seem to be true at all. In fact, the oft-quoted counterexample is the egg falling off a countertop; we often see eggs fall and break into a large number of pieces, but we never see a broken egg rise and reassemble into a whole egg on the countertop. This is due to the second law of thermodynamics, which asserts that entropy cannot decrease globally. However, it is currently thought that we cannot conclude that T-symmetry is violated based on thermodynamics, as the fact that entropy always increases is based on boundary conditions (the state of the universe at the time-boundary $t = 0$) and not physical laws themselves [9]. In other words, the second law of thermodynamics is more of a statement that the initial state of the universe was extremely ordered and thus things could only get messier. At the particle physics level, one might think that all interactions are completely reversible, which is true to a certain extent. However, it has been shown that T-symmetry is indeed violated in a few rare subatomic interactions.

The fact that these three discrete near-symmetries are not complete symmetries has huge implications for cosmology and physics as a whole [10]. In fact, there is currently a consensus that this so-called CP-violation is necessary in order for the universe to have more matter than antimatter - one of the most important unsolved problems in physics. Unfortunately, there are currently no known theoretical and/or mathematical mechanisms from which CP-violation arises in the early universe, which is yet another indicator of new physics - physics beyond our familiar Standard Model.


